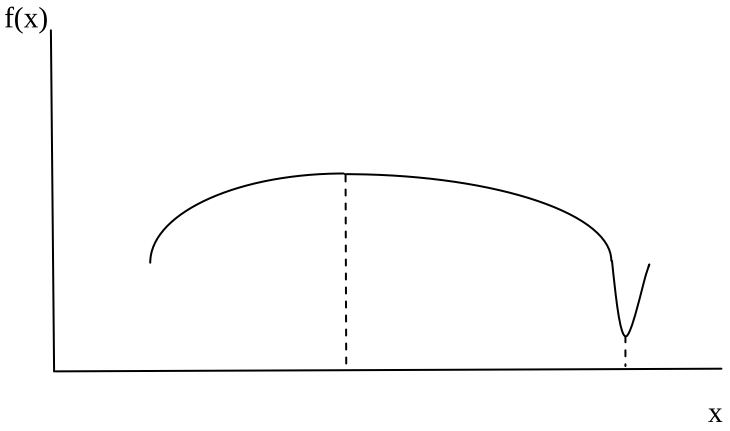
Lecture 4: Optimization

4.1 Maximizing or Minimizing a Function of a Single Variable

1. Given a real valued function, $y = f(x)$ we will be concerned with the existence of extreme values of the dependent variable y and the values of x which generate these extrema. (maxima or minima) The function f(x) is called the *objective function* and the independent variable x is called the *choice variable*. The problem of finding the value or the set of values of the choice variable which yield extrema of the objective function is called optimization. In order to avoid boundary optima, we will assume that $f : X \to R$, where X is an open interval of R. All of the optima characterized will be termed interior optima.

2. Definition. f has a local maximum (minimum) at a point x', if for all x in an open interval $(x' - \mu, x' + \mu)$, we have that $f(x') > (<) f(x)$.

3. In the figure below, we illustrate local maxima and minima.



4. *Proposition 1.* Let f be twice differentiable. Let there exists an $x^o \in X$ such that $f'(x^o) = 0$.

(i) If $f''(x^o) < 0$, then f has a local maximum at x^o. If, in addition, $f'' < 0$ for all x or if f is strictly concave, then the local maximum is a unique global maximum.

(ii) If $f''(x^o) > 0$, then f has a local minimum at x^o. If, in addition, $f'' > 0$ for all x or if f is strictly convex, then the local minimum is a unique global minimum.

The zero derivative condition is called the *first order condition* and the second derivative condition is called the *second order condition*.

5. Examples

#1 Let $f = ax - bx^2$, $a,b > 0$ and $x > 0$. Find a maximum.

Here, $f' = 0$ implies $x' = a/2b$. Moreover, $f'' = -2b < 0$, for all x. Thus, we have a global maximum.

#2 Let $f = x + x^{-1}$, where $x > 0$. Find a minimum.

Here, $f' = 0$ implies that $x^{-2} = 1$, so that $x' = 1$. In this case, $f'' = 2x^{-3} > 0$. Thus, we have a global minimum.

4.2 Maximizing or Minimizing a Function of Many Variables.

1. We consider a differentiable function of many variables $y = f(x_1,...,x_n)$. This function has a local maximum (minimum) at a point $x' = (x_1,...,x_n)$, if the values of the function are greater than (less than) image values of the function in a neighborhood of x'. The domain of f is thought of as a subset X of R^n, where each point of X has a neighborhood of points surrounding it which belongs to X. We will characterize interior optima.

2. We have the following result.

Proposition 2. If a differentiable function f has a maximum or a minimum at $x' \in X$, then $f_i(x^o) = 0$, for all i.

This condition states that at a maximum or a minimum, all partial derivatives are zero. This depicts the top of a hill or a bottom of a valley. Operationally, the n partial derivative functions set equal to zero give us n equations in n unknowns to be solved for the extreme point x'. These conditions are called the first order conditions (FOC).

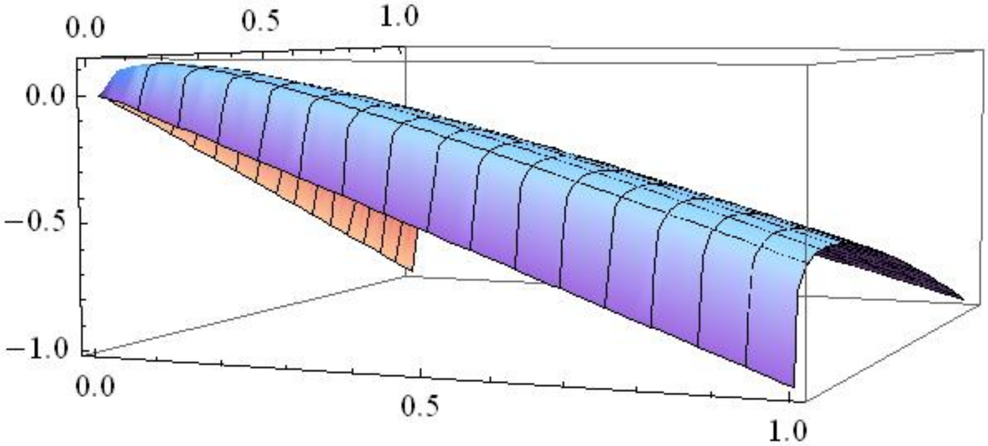
3. Example

Find the maximum of $(x_1x_2)^{1/4} - x_1 - x_{2.}$ Computing, we have

$.25(x_1)^{-.75}(x_2)^{.25} - 1 = 0$

$.25(x_2)^{-.75}(x_1)^{.25} - 1 = 0$

These imply that $x_i = x$ and that $x_i = 1/16$. At optimum, we have that y = .125. The figure below illustrates.

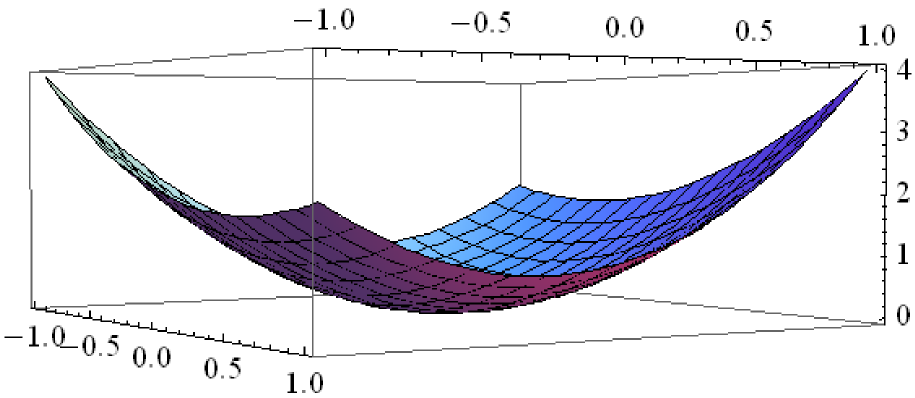


#2 Find the minimum of $z = x^2 + xy + 2y^2$. The FOC are

$2x + y = 0,$

$x + 4y = 0.$

Solving for the critical values x = 0 and y = 0. The illustration follows.



4.3 Constrained Optimization

1. One of the most common problems in economics involves maximizing or minimizing a function subject to a constraint. The cost minimization problem subject to an output constraint is an example. We are again interested in characterizing interior or non-boundary constrained optima.

2. The basic problem is to maximize a function of at least two independent variables subject to a constraint. We write the objective function as $f(x_1,...,x_n)$and the constraint as $g(x_1,...,x_n) = 0$. The constraint set is written as $C = \{ x \mid g(x_1,...,x_n) = 0\}$. The function f maps a subset of R^n into the real line. We write the problem as

$$\underset{\{x_1,...,x_n\}}{\text{Max}}\ f(x_1,...,x_n) \text{ subject to } g(x_1,...,x_n) = 0.$$

3. This problem has the following solution. Let $x = (x_1,...,x_n)$.

Proposition 3. Let f be a differentiable function whose n independent variables are restricted by the differentiable constraint $g(x) = 0$. Form the function $L(\lambda,x) \equiv f(x) + \lambda g(x)$, where λ is an undetermined multiplier. If x^o is an interior maximizer or minimizer of f subject to $g(x) = 0$, then there is a λ^o such that

(1) $\partial L(\lambda^o, x^o)/\partial x_i = 0$, for all i, and

(2) $\partial L(\lambda^o, x^o)/\partial \lambda = 0$.

Remark: L is the Lagrangian function and λ is the Lagrangian multiplier. Conditions (1) and (2) are again called the FOC. They constitute n + 1 equations in n + 1 unknowns.

4. Consider the cost minimization problem from Lecture 2.

$$\underset{\{x_1,x_2\}}{\text{Min}}\ (p_1x_1 + p_2x_2) \text{ subject to } q^t = f(x_1,x_2).$$

Forming the Lagrangian, we have

$L = (p_1x_1 + p_2x_2) + \lambda[q^t - f(x_1,x_2)]$.

The relevant FOC are

(1) $L_\lambda = q^t - f(x_1,x_2) = 0$,

(2) $L_1 = p_1 - \lambda f_1 = 0$,

(3) $L_2 = p_2 - \lambda f_2 = 0$.

Condition (1) just says that the firm must obey its output constraint and conditions (2) - (3) say that

$$\frac{p_1}{p_2} = \frac{f_1}{f_2}.$$

That is, the MRS should equal the price ratio.

The dual to this problem is the maximization of output subject to a cost constraint.

$$\max_{\{x_1,x_2\}} f(x_1,x_2) \text{ subject to } c - p_1x_1 - p_2x_2 = 0.$$

We have that

$L = f(x_1,x_2) + \lambda(c - p_1x_1 - p_2x_2)$.

The FOC now lead to

(1) $L_\lambda = c - p_1x_1 - p_2x_2 = 0$

(2) $f_1 - \lambda p_1 = 0$,

(3) $f_2 - \lambda p_2 = 0$.

We arrive at the same MRS = price ratio condition and that the output constraint must be met.

5. Numerical Example

Let $f = x_1x_2$ and let $p_1 = 2$ and $p_2 = 2$. Solve the cost minimization problem with a target output of 16. The Lagrangian is

$L = 2x_1 + 2x_2 +\lambda(16 -x_1x_2)$.

The FOC are given by

(1) $x_1x_2 - 16 = 0$,

(2) $2 - \lambda x_2 = 0$,

(3) $2 - \lambda x_1 = 0$.

Clearly $x_i = x$, so that using (1), $x^2 =16$ and $x_i = 4$.