The Elastic Pi Problem

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1 The problem

Two balls of mass m_1 and m_2 are beside a wall. Mass m_1 is positioned between m_2 and the wall and is at rest. Mass m_2 is moving with velocity v towards m_1, and all collisions (both ball-ball and ball-wall) are elastic. If $m_2 = m_1$, there are 3 collisions before m_2 escapes; if $m_2 = 100m_1$, there are 31 collisions; if $m_2 = 100^2 m_1$, there are 314; and if $m_2 = 100^3 m_1$, there are 3141. In other words, if $m_2 = 100^n m_1$, then m_2 escapes after $\lfloor 10^n \pi \rfloor$ collisions. Why π?

2 Verbal description of what happens

When m_2 hits m_1, it transfers some of its momentum (in the wall direction) to m_1. This sets m_1 in motion towards the wall. Upon hitting the wall, m_1 reverses direction (maintaining the same speed since the collision is elastic). It alternates between hitting m_2 and the wall. Every time m_1 hits m_2, m_2 loses velocity in the wall direction, until eventually it is moving away from the wall fast enough that m_1 cannot catch up with it. At this point, m_2 escapes off to infinity, and no more collisions occur.

3 A Preliminary Result

We begin by establishing a general result about the change in velocity when two moving masses collide.

Claim 1. *Suppose two masses, m_1 and m_1, collide with velocities v_1 and v_2. Then after the collision their velocities are*

$$\tilde{v}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$
$$\tilde{v}_2 = \left(\frac{2m_1}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2.$$

Figure 1: Schematic of situation.

Proof. The center of mass of the two balls is moving at velocity

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2},$$

so in the reference frame of the center of mass, the balls are moving at velocities $v_1 - v_{\mathrm{CM}}$ and $v_2 - v_{\mathrm{CM}}$. In this frame, the total momentum is zero and thus their momentums are equal and opposite. After the collision, by conservation of momentum, they must have equal and opposite momenta and since the collision is elastic, they must be traveling at the same speeds (but with directions reversed). Therefore in the center of mass frame, after the collision the balls are traveling with velocities $v_{\mathrm{CM}} - v_1$ and $v_{\mathrm{CM}} - v_2$ respectively. In the reference frame of the wall, their final velocities are therefore $\tilde{v}_1 = (v_{\mathrm{CM}} - v_1) + v_{\mathrm{CM}}$ and $\tilde{v}_2 = (v_{\mathrm{CM}} - v_2) + v_{\mathrm{CM}}$. Thus,

$$\tilde{v}_1 = 2v_{\mathrm{CM}} - v_1 = \frac{2(m_1 v_1 + m_2 v_2) - (m_1 + m_2)v_1}{m_1 + m_2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$
$$\tilde{v}_2 = 2v_{\mathrm{CM}} - v_2 = \frac{2(m_1 v_1 + m_2 v_2) - (m_1 + m_2)v_2}{m_1 + m_2} = \left(\frac{2m_1}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2.$$

□

4 Calculating the velocity as a function of number of collisions

We detail in this section how the velocities change after each collision. Let $v_1^{(t)}$ and $v_2^{(t)}$ denote the velocities of the balls after t collisions. To start we have $v_1^{(0)} = 0$ and $v_1^{(0)} = -v$ (we define away from the wall to be positive). The system alternates between two kinds of collisions

(1) **Ball-Ball:** By Claim 1, we have

$$v_1^{(t+1)} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1^{(t)} + \left(\frac{2m_2}{m_1 + m_2}\right) v_2^{(t)}$$
$$v_2^{(t+1)} = \left(\frac{2m_1}{m_1 + m_2}\right) v_1^{(t)} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2^{(t)}.$$

(2) **Ball-Wall:** When m_1 collides with the wall, it changes direction, but doesn't change speed (since the collision is elastic):

$$v_1^{(t+2)} = -v_1^{(t+1)}$$
$$v_2^{(t+2)} = v_2^{(t+1)}.$$

Putting this together, we see that

$$v_1^{(t+2)} = -\left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1^{(t)} - \left(\frac{2m_2}{m_1 + m_2}\right) v_2^{(t)}$$
$$v_2^{(t+2)} = \left(\frac{2m_1}{m_1 + m_2}\right) v_1^{(t)} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2^{(t)}.$$

Define $R = m_2/m_1$. Then this can be written as

$$v_1^{(t+2)} = \left(\frac{R-1}{R+1}\right) v_1^{(t)} - \left(\frac{2R}{R+1}\right) v_2^{(t)}$$
$$v_2^{(t+2)} = \left(\frac{2}{R+1}\right) v_1^{(t)} + \left(\frac{R-1}{R+1}\right) v_2^{(t)}.$$

This can be expressed more compactly in vector/matrix form as

$$\mathbf{v}^{(t+2)} = \mathbf{A}\mathbf{v}^{(t)}, \tag{1}$$

where

$$\mathbf{A} = \begin{pmatrix} (R-1)/(R+1) & -2R/(R+1) \\ 2/(R+1) & (R-1)/(R+1) \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}^{(t)} = \begin{pmatrix} v_1^{(t)} \\ v_2^{(t)} \end{pmatrix}$$

.

By Equation (1),

$$\mathbf{v}^{(2t)} = \mathbf{A}\mathbf{v}^{(2t-2)} = \mathbf{A}^2\mathbf{v}^{(2t-4)} = \cdots = \mathbf{A}^t\mathbf{v}^{(0)}.$$

Thus, after $2t$ collisions, the balls will have velocities given by

$$\mathbf{A}^t \begin{pmatrix} 0 \\ -v \end{pmatrix}$$

Our goal is to calculate the total number of collisions. Notice that the balls will continue to collide as long as $|v_1| \geq v_2$ (since this means that m_1 is able to overtake m_2). Thus, after the last collision, $|v_1| < v_2$ for the first time. With this equation for (v_1, v_2) as a function of the number of collisions, we can compute

$$T = \min\{t : |v_1^{(t)}| < v_2^{(t)}\}.$$

And doing so for $m_2/m_1 = 1, 100, 100^2, \ldots, 100^{10}$ we confirm that $T = 3, 31, 314, \ldots, 31415926535$.

5 A Useful Geometric View

We know that initially $v_1 = 0 > -v = v_2$ and that through the course of collisions, the "path" in $v_1 - v_2$ space eventually crosses the $v_2 = |v_1|$ boundary. Based on the equation for $\mathbf{v}^{(t)}$ above, we can numerically calculate the path in $v_1 - v_2$ space for any fixed $R = m_2/m_1$ (see Figure 2). A striking feature of these plots is that the vectors $\mathbf{v}^{(t)}$ always lie on an ellipse with major axis along the v_1 axis (and a factor of $\sqrt{R}$ of the minor axis).

6 Why an ellipse?

Since all collisions are elastic, kinetic energy is conserved and therefore

$$K = \frac{1}{2}m_1 v_1^{(t)2} + \frac{1}{2}m_2 v_2^{(t)2}.$$

And initially $K = \frac{1}{2}m_2 v^2$ (where the one disadvantage of this choice is that the units look wrong). Since the magnitude of v does not matter, we can take it to be 1 so that $K = m_2/2$. This is advantageous because then the above equation can be written more simply as

$$1 = v_1^{(t)2}/R + v_2^{(t)2}$$

which is the equation for the ellipse observed in the figure.

7 From ellipse to circle

Observing that $\mathbf{v}^{(t)}$ is confined to an ellipse, we make things simpler by transforming to a coordinate system in which this is a circle. Let $u_1 = v_1/\sqrt{R}$ and $u_2 = v_2$. Then, $1 = u_1^{(t)2} + u_2^{(t)2}$. We can express the recurrence equation for $\mathbf{v}^{(t)}$ in terms of $\mathbf{u}^{(t)}$. Since $\mathbf{u}^{(t)}$ always lies on a circle, the matrix which gives $\mathbf{u}^{(t)}$ must be simply a rotation matrix.

From Equation (1), we have

$$\begin{pmatrix} \sqrt{R} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}^{(t+2)} = \mathbf{v}^{(t+2)} = \mathbf{A}\mathbf{v}^{(t)} = \mathbf{A} \begin{pmatrix} \sqrt{R} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}^{(t)}$$

or

$$\begin{aligned}
\mathbf{u}^{(t+2)} &= \begin{pmatrix} 1/\sqrt{R} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (R-1)/(R+1) & -2R/(R+1) \\ 2/(R+1) & (R-1)/(R+1) \end{pmatrix} \begin{pmatrix} \sqrt{R} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}^{(t)} \\
&= \begin{pmatrix} 1/\sqrt{R} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{R}(R-1)/(R+1) & -2R/(R+1) \\ 2\sqrt{R}/(R+1) & (R-1)/(R+1) \end{pmatrix} \mathbf{u}^{(t)} \\
&= \begin{pmatrix} (R-1)/(R+1) & -2\sqrt{R}/(R+1) \\ 2\sqrt{R}/(R+1) & (R-1)/(R+1) \end{pmatrix} \mathbf{u}^{(t)}
\end{aligned}$$

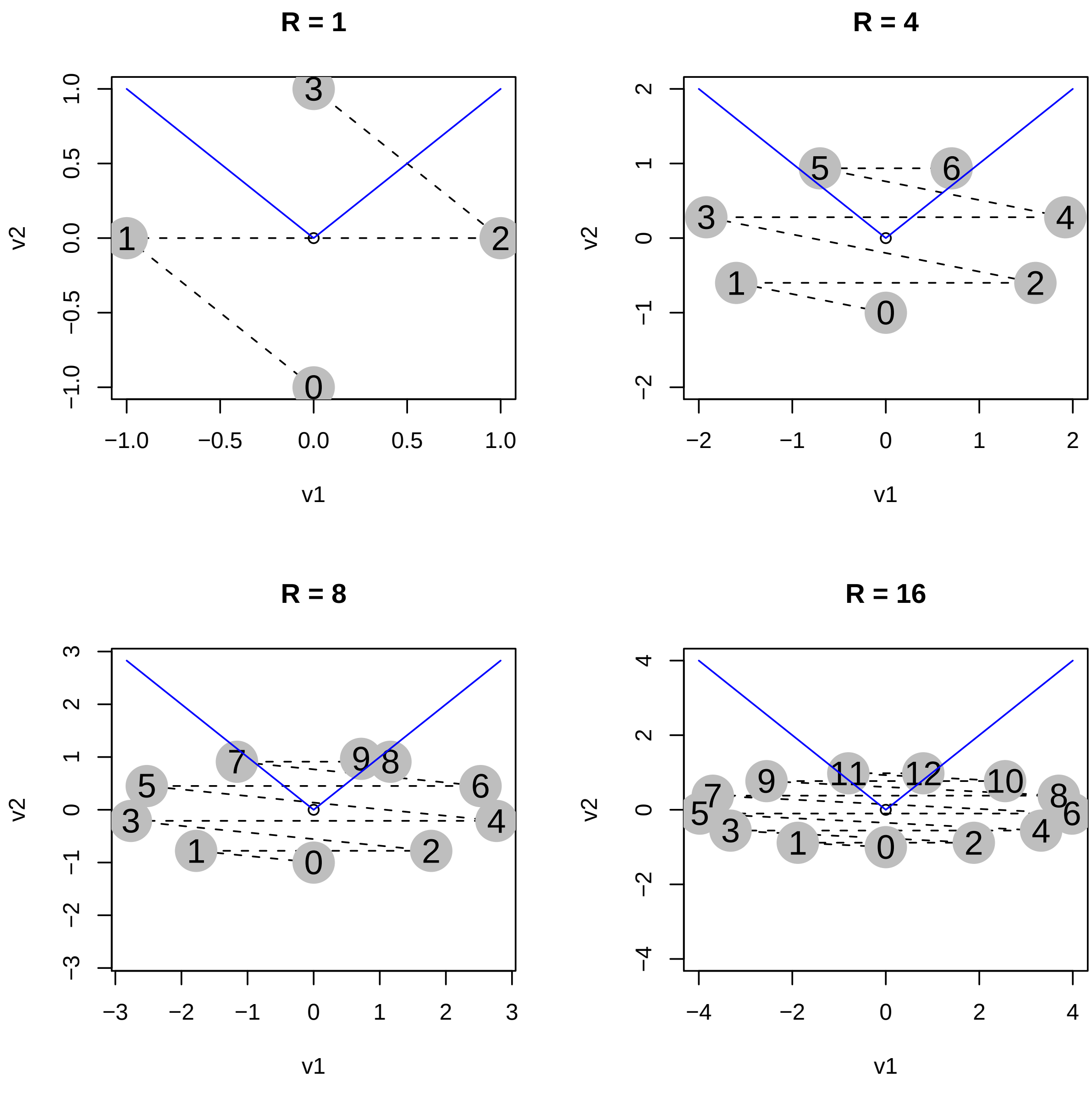

Figure 2: $v_2^{(t)}$ as a function of $v_1^{(t)}$.

We claim that this is a rotation matrix. Indeed, observing that

$$\left(\frac{R-1}{R+1}\right)^2 + \left(\frac{2\sqrt{R}}{R+1}\right)^2 = \frac{R^2-2R+1+4R}{(R+1)^2} = 1,$$

we can write the above recurrence in terms of $\theta = \arcsin\left(\frac{2\sqrt{R}}{R+1}\right)$:

$$\mathbf{u}^{(t+2)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{u}^{(t)}.$$

Multiplying a vector by this matrix rotates it by theta (counterclockwise). This observation is useful since t rotations by θ is identical to a single rotation by $t\theta$. That is,

$$\mathbf{u}^{(2t)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^t \mathbf{u}^{(0)} = \begin{pmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{pmatrix} \mathbf{u}^{(0)}.$$

Finally, since $\mathbf{u}^{(0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, it follows that

$$\mathbf{u}^{(2t)} = \begin{pmatrix} \sin\theta t \\ -\cos\theta t \end{pmatrix}.$$

This gives an expression for $\mathbf{v}^{(2t)}$ that conveniently does not require t matrix multiplications:

$$\mathbf{v}^{(2t)} = \begin{pmatrix} \sqrt{R}\sin\theta t \\ -\cos\theta t \end{pmatrix}.$$

8 Answering the question... at long last

$$\begin{aligned}
T &= \min\{t : |v_1^{(t)}| < v_2^{(t)}\} \\
&= \min\{t : |\sqrt{R}\sin\theta t/2| < -\cos\theta t/2 \text{ and } \cos\theta t/2 < 0\} \\
&= \min\{t : -\tan\theta t/2 < 1/\sqrt{R}\} \\
&= \min\{t : \tan(\pi - \theta t/2) < 1/\sqrt{R}\} \\
&= \min\{t : \pi - \theta t/2 < \arctan(1/\sqrt{R})\} \\
&= \min\{t : t > (2/\theta)(\pi - \arctan(1/\sqrt{R}))\} \\
&= \lceil (2/\theta)(\pi - \arctan(1/\sqrt{R})) \rceil
\end{aligned}$$

That is,

$$T = \left\lceil \frac{2\pi - 2\arctan(1/\sqrt{R})}{\arcsin\left(\frac{2\sqrt{R}}{R+1}\right)} \right\rceil$$

Now, for small x, $\sin x \approx x$ and $\tan x \approx x$. Thus, for large R,

$$T \approx \left\lceil \frac{2\pi - 2/\sqrt{R}}{\frac{2\sqrt{R}}{R+1}} \right\rceil \approx \left\lceil \frac{2\pi - 2/\sqrt{R}}{\frac{2}{\sqrt{R}}} \right\rceil = \left\lceil \pi\sqrt{R} - 1 \right\rceil = \left\lfloor \pi\sqrt{R} \right\rfloor$$